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A quasistatic model of the evolution of an interface inside a deformed solid☆

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Abstract

A one-dimensional integral equation, the solution of which enables one to follow the (small and continuous) change in the form of the interface as a function of a time-like loading parameter, is derived by constructing a formal trinomial asymptotic form of the elastic fields. The operator and other data of the equation are expressed in terms of the Steklov-Poincaré operators for separated phases at the initial instant and the solutions of the problem with a fixed interface. An investigation of the equation establishes the stability of the development and the possibility of bifurcations or the need to take dynamic effects into account. A well-known thermodynamic condition at the interface and a new condition of its classical stability are obtained as a special case.

1. Formulation of the quasistatic problem

Consider a plane elastic body Ω, the internal part of which Ω_t^+ has undergone a phase transition. We will assume that the boundary Γ_t between the two phases is a simple smooth closed contour, having no common points with the boundary $\partial\Omega$ of the body itself. Here Ω_t^+ is the region bounded by the contour Γ_t, and $\Omega_t^- = \Omega\backslash(\Omega_t^+ \cup \Gamma_t)$. To simplify the formulation of the problem we will assume that the body is rigidly clamped along the non-empty open arc $\gamma \subset \partial\Omega$. The following external forces are applied to the remaining part Σ of the boundary $\partial\Omega$

$$g^t(x) = g^0(x) + tg^1(x) + t^2 g^2(x) + \cdots \tag{1.1}$$

Here t is a dimensionless time-like parameter of the loading, non-negative and monotonic with respect to real time. The rate of change of the parameter t is assumed to be small compared with the propagation velocities of elastic waves, referred to the characteristic dimension of the composite body. The latter fact enables us justifiably to neglect the inertial terms and ensures a quasistatic formulation of the problem of the evolution of the interface, in which, for a known initial contour Γ_0, it is required to obtain its position Γ_t for $t > 0$.

For each $t \geq 0$ the displacement vector u^t, the strain tensor ε^t and the stress tensor σ^t, connected by the linear relations

$$\varepsilon_{jk}^{t\pm} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}u_k^{t\pm} + \frac{\partial}{\partial x_k}u_j^{t\pm}\right), \quad \sigma^{t\pm} = A^{\pm}\varepsilon^{t\pm} \text{ in } \Omega_t^{\pm} \tag{1.2}$$

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satisfy homogeneous equilibrium equations (there are no mass forces), and also the boundary conditions on the external boundary and the matching conditions (ideal contact) at the interface

$$-\frac{\partial}{\partial x_j}\sigma_{kj}^{t\pm} = 0, \quad k = 1,2, \quad \text{in} \quad \Omega_t^{\pm} \tag{1.3}$$

$$n_j\sigma_{kj}^t = g_k^t, \quad k = 1,2, \quad \text{on} \quad \Sigma, \quad u^{t-} = 0 \quad \text{on} \quad \Upsilon \tag{1.4}$$

$$[u] = 0, \quad \nu_j^t\sigma_{kj}^{t+} = \nu_j^t\sigma_{kj}^{t-} \quad \text{on} \quad \Gamma_t \tag{1.5}$$

Moreover $A(x)=A^{\pm}$ are Hooke tensors for the phases $\Omega_t^{\pm}$, summation is carried out over repeated indices $j=1, 2$, $[v]=v^+-v^-$ is the jump in the function v on Γ_t, $\nu^t=(\nu_1^t, \nu_2^t)$ and $n=(n_1, n_2)$ are the unit vectors of the outward normals on Γ_t (relative to the regions enveloped by the contours) and $\partial\Omega \cup \Gamma_0$, in particular $\nu^0=n$ on Γ_0. In view of the presence in (1.4) of the Dirichlet conditions, problem (1.3)–(1.5) is uniquely solvable for all $t\geq 0$ and $g^t \in L_2(\Sigma)^2$ irrespective of the dependence on the position of the interface (the condition imposed on the vector function g^t may be reduced). The solution $u^t \in H^1(\Omega)^2$ turns out to be infinitely differentiable inside the regions $\Omega_t^{\pm}$, and for a continuous right-hand side of the first condition (1.4) also on the open arc Σ but not at its end points. On the continuous contour Γ_t there are unilateral derivatives of the displacements u^t of any order.

The position and shape of the contour Γ_t is determined from the following requirement (compare with Refs. 1–3 etc.): at any instant $t\geq 0$ the functional

$$\mathcal{U}_t = U_t + \sum_{\pm}\gamma_t^{\pm}\operatorname{mes}_2\Omega_t^{\pm} \tag{1.6}$$

calculated when solving problem (1.3)–(1.5), takes the least value compared with the other possible positions of the interface. In equality (1.6) U_t is the potential strain energy, stored by the composite body $\Omega_t{:=}\Omega_t^+ \cup \Omega_t^-$,

$$U_t = E_t - R_t, \quad E_t = \frac{1}{2}\sum_{\pm}\int_{\Omega_t^{\pm}}\sigma^{t\pm} : \varepsilon^{t\pm}dx, \quad R_t = \int_{\Sigma}u^{t-}\cdot g^t ds \tag{1.7}$$

$$\gamma_t^{\pm} = \gamma_0^{\pm} + t\gamma_1^{\pm} + t^2\gamma_2^{\pm} + \cdots \tag{1.8}$$

Moreover, $\operatorname{mes}_2\Omega_t^{\pm}$ is the area of the figure $\Omega_t^{\pm}$, E_t is the elastic energy, R_t is the work done by external forces and $\gamma_t^{\pm}$ is the energy density of the unstrained state (usually $\gamma_1^{\pm}=\gamma_2^{\pm}=0$). In formula (1.7) a scalar product of vectors is denoted by a dot and the convolution of tensors is denoted by a colon.

In this paper the complex mathematical problem of simultaneously finding the interface and elastic field (the existence and uniqueness of the solution and the smoothness of the boundaries and fields) is not touched upon (in this connection see, for example, Ref. 3). It is assumed that the solution at the initial instant $t=0$ is known and possesses the required properties. As a result of formal asymptotic analysis, functional (1.6) and differential problem (1.3)–(1.5) are replaced by an asymptotic approximation to $\mathcal{U}_t$ and the corresponding Euler equation, which generate an integral operator on the contour Γ_0. The latter is also the object of the investigation. This also applies to the problem with surface energy mentioned in Section 7, where an additional integral over the arc Γ_t is introduced into functional (1.6). The main assumptions for which these substitutions generally make sense are smoothness, closure and simplicity of the contour separating the phases. Note that in the case of continuous test functions, the first and second variations of the functional (1.6) itself are identical with the analogous variations of its asymptotic approximation.

In Sections 3 and 4 we construct three terms of the asymptotic form of the solution of problem (3)–(5) with respect to the parameter t and the function h, unknown in advance, which specify the perturbation of the contour Γ_0 (see later formulae (2.1) and (2.5)). As in the problem of the propagation of a plane crack in a brittle elastic solid within the framework of the Griffith energy criterion (see Refs. 4–6), the quasistatic model arises as a result of replacing functional (1.6) by its quadratic approximation

$$\mathcal{U}^{(0)} + t\mathcal{U}^{(1)}(h) + t^2\mathcal{U}^{(2)}(h) \tag{1.9}$$

obtained using the asymptotic form mentioned above. The second term $\mathcal{U}^{(1)}(h)$ depends linearly on h, and hence functional (1.9) may reach a minimum for a small variation of the contour Γ_0 only in the case of the relation (3.8) between the fields $\sigma^{0\pm} = \sigma^{t\pm}|_{t=0}$ and $\varepsilon^{0\pm} = \varepsilon^{t\pm}|_{t=0}$ on Γ_0. This relation, known as the thermodynamic condition at the interface (see Refs. 1–3 etc.), is the necessary condition for functional (1.6) to have a minimum at the point $t=0$ for the separating line Γ_0 and is interpreted in Section 7 as equality of the jumps in the densities of the surface enthalpy and the residual internal energy.

The last term in trinomial (1.9) is a quadratic functional of h, and the condition for it to have an extremum produces, for the function h, the growth Eq. (4.10) on Γ_0, so called by analogy with the equation describing the growth of the free surface of a quasi-statically developing crack in a brittle elastic three-dimensional solid (see Refs. 4–6). We emphasise that the data of the above-mentioned equation is found from the solutions u^0 and $\hat{u}^1$ of problem (1.3)–(1.5) for the composite body Ω_0 for loads g^0 and g^1, while the pseudo-differential operator B occurring in it is expressed in terms of the Steklov-Poincaré operators for disconnected bodies $\Omega_0^{\pm}$ and differential operations on the arc Γ_0 (see Sections 5 and 2). As might have been expected, the terms t^2g^2 and $t^2\gamma_2^{\pm}$ of representations (1.1) and (1.8) do not participate in the formation of the growth equation.

From the solution $h = t\hat{h}$ of the equation one can approximately determine the shape of the contour Γ_t, or, more accurately, the initial velocities $\hat{h}(s)$ with which the points $s \in \Gamma_0$ move along the normal directions. The error is $O(t^3)$, and if it is desired to follow the development of the interface over a large range of variation of the time-like parameter, one must use the proposed asymptotic constructions in a step-by-step mode (similar to the Peano method): the interval $(0, T)$ is divided into small intervals t_{N-1}, t_N and, using the solution h_{N-1} one determines the position of the interface Γ_N at the instant $t=t_N$, at which new data on the growth equation are set up for the function h_N.

In Section 6 we discuss the stability of the quasistatic development of the boundary Γ_t within the framework of the proposed asymptotic model. The conclusions reached on the basis of the asymptotic formulae from Sections 3 and 4 differ from those reached by other researchers (Refs. 7,8 etc.), based on an analysis of the stability of the equation of the evolution of the interface, related to the attendant physical processes. If the parameter t is removed and it is assumed that, in formulae (1.1) and (1.8),

$$g^t = g^0, \quad \gamma_{\pm}^t = \gamma_{\pm}^0 \tag{1.10}$$

the calculations will retain their meaning, allowing of the first and second variations of the functional (1.6) when the interface is perturbed. The formula for the first variation does not differ from the existing ones (see Refs. 1–3 etc.), but the expression for the second variation is simpler and clearer compared with that obtained previously,[9,10] taking into account the material derivatives of the functionals – the classical apparatus of the theory of shape optimization (see Refs. 11,12 etc.).

2. Local coordinates

In the neighbourhood V of the line Γ_0 we introduce a system of natural curvilinear coordinates (n, s), where n is the distance to Γ_0 and $\mp n > 0$ in $\Omega_0^{\pm} \cap V$, while s is the length of the arc on Γ_0. For small $t \geq 0$ the contour Γ_t is defined by the following equality, in which $h(t;\cdot)$ is a continuous function on the contour Γ_0

$$n = h(t; s) \tag{2.1}$$

Since the differential operator of the gradient in (n, s) coordinates has the form $(\partial_n, D(n, s)^{-1}\partial_s)$, the normal vector ν and the tangential vector τ on Γ_t are calculated from the formulae

$$\begin{gathered} d(n,s)\nu(t;s) = (1, -D(h(t;s),s)^{-1}h'(t;s)), \quad d(n,s)\tau(t;s) = (D(h(t;s),s)^{-1}h'(t;s), 1) \\ \partial_n = \partial/\partial n, \quad \partial_s = \partial/\partial s, \quad h' = \partial_s h \end{gathered} \tag{2.2}$$

Here $D(n, s) = 1 + n\kappa(s)$ is the Jacobian, $d = (1 + h'^2 D^2)^{1/2}$ is a normalizing factor and $\kappa(s)$ is the curvature of the arc Γ_0 at the point s. On the right-hand sides of equalities (2.2) the first term represents the projection of the vector onto the direction n while the second term represents the projection onto the direction s. The components of the strain tensor and the equilibrium equations in curvilinear coordinates are as follows:

$$\varepsilon_{nn} = \partial_n u_n, \quad \varepsilon_{ss} = D^{-1}(\partial_s u_s + \kappa u_n), \quad \varepsilon_{ns} = \varepsilon_{sn} = (\partial_n u_s + D^{-1}(\partial_s u_n - \kappa u_s))/2 \tag{2.3}$$

$$\partial_n\sigma_{nn} + D^{-1}(\partial_s\sigma_{ns} - \kappa(\sigma_{ss} - \sigma_{nn})) = 0, \quad \partial_n\sigma_{sn} + D^{-1}(\partial_s\sigma_{ss} + 2\kappa\sigma_{sn})) = 0 \tag{2.4}$$

We will assign the functions h and their derivatives with respect to s the order t, i.e. we will use the relation

$$h(t;\, s) = t\hat{h}(s) + \cdots \tag{2.5}$$

The quantity $O(t^2)$ in relations (2.5) is ignored in accordance with the assumption in Section 1 that the terms t^2g^2 and $t^2\gamma_2^{\pm}$ from expansions (1.1) and (1.8) have no effect on the growth equation of interest. We will seek a solution of problem (1.3)–(1.5) in the composite body Ω_t with a regularly perturbed separation line in the form of the asymptotic series

$$u^t(x) = u^0(x) + u^1(t;\, x) + u^2(t;\, x) + \cdots \tag{2.6}$$

We will assign the term $u^p(t;x)$ the order t^p ($p = 0, 1, 2, \ldots$). For brevity the argument t of the functions (2.1) and (2.6) will henceforth be omitted.

We will remove the conditions, imposed on the basic contour Γ_0, that there should be no jumps in the displacements on the perturbed contour Γ_t. To do this we will smoothly extend the field $u^{p\pm}$ from the regions $\Omega_0^{\pm}$ to $\Omega_0^{\mp} \cap V$ and expand them in Taylor series with respect to the variable n, which, in agreement with formula (2.1), we take equal to $h(t;s)$. As a result we obtain that on the arc Γ_0 the following relation is satisfied with accuracy $O(t^3)$

$$[u^t] = [u^0] + \{[u^1] + h[\partial_n u^0]\} + \left\{[u^2] + h[\partial_n u^1] + \frac{1}{2}h^2[\partial_n^2 u^0]\right\} + \cdots \tag{2.7}$$

For similar processing of the jumps in the stresses we note that, according to expression (2.2), on the contour Γ_t the following equalities hold

$$d^2\sigma_{\nu\nu} = \sigma_{nn} - 2h'D^{-1}\sigma_{ns} + h'^2D^{-2}\sigma_{ss}, \quad d^2\sigma_{\nu\tau} = \sigma_{ns} + h'D^{-1}(\sigma_{nn} - \sigma_{ss}) + h'^2D^{-2}\sigma_{ns}$$

Hence

$$\begin{aligned} d^2[\sigma^t_{\nu\nu}] &= [\sigma^0_{nn}] + \{[\sigma^1_{nn}] + h[\partial_n\sigma^0_{nn}] - 2h'[\sigma^0_{ns}]\} + \\ &+ \left\{[\sigma^2_{nn}] + h[\partial_n\sigma^1_{nn}] - 2h'[\sigma^1_{ns}] + \frac{1}{2}h^2[\partial_n^2\sigma^0_{nn}] - 2h'h([\partial_n\sigma^0_{ns}] - \kappa[\sigma^0_{ns}]) + h'^2[\sigma^0_{ss}]\right\} + \cdots \end{aligned} \tag{2.8}$$

$$\begin{aligned} d^2[\sigma^t_{\nu\tau}] &= [\sigma^0_{ns}] + \{[\sigma^1_{ns}] + h[\partial_n\sigma^0_{ns}] + h'([\sigma^0_{nn}] - [\sigma^0_{ss}])\} + \\ &+ \left\{[\sigma^2_{ns}] + h[\partial_n\sigma^1_{ns}] + h'([\sigma^1_{nn}] - [\sigma^1_{ss}]) + \frac{1}{2}h^2[\partial_n^2\sigma^0_{ns}] + \right. \\ &\left. + h'h([\partial_n\sigma^0_{nn}] - [\partial_n\sigma^0_{ss}]) - h'h\kappa([\sigma^0_{nn}] - [\sigma^0_{ss}]) + h'^2[\sigma^0_{ns}]\right\} + \cdots \end{aligned} \tag{2.9}$$

Here $\sigma^p_{\alpha\beta}$ are the stresses calculated on the displacements u^p. We emphasise that, thanks to relation (2.5), on the right-hand sides of equalities (2.7)–(2.9) the first braces enclose quantities $O(t)$ while the second braces enclose quantities $O(t^2)$. Finally, the following expansions of the functionals from definition (1.6) correspond to ansatzes (2.6) and (1.1), (1.8)

$$\begin{aligned} U_t &= -\frac{1}{2}R_t = -\frac{1}{2}\int_\Sigma g^0\cdot u^0 ds - \frac{1}{2}\int_\Sigma (g^0\cdot u^1 + tg^1\cdot u^0)ds - \\ &-\frac{1}{2}\int_\Sigma (g^0\cdot u^2 + tg^1\cdot u^1 + t^2g^2\cdot u^0)ds + \cdots \end{aligned} \tag{2.10}$$

$$\sum_{\pm}\gamma_t^{\pm}\mathrm{mes}_2\Omega_t^{\pm} = \sum_{\pm}\gamma_t^{\pm}\mathrm{mes}_2\Omega_0^{\pm} + [\gamma_t]\int_{\Gamma_0}\int_0^{h(t;\,s)} D(n,s)\,dn\,ds =$$
$$= \sum_{\pm}\gamma_0^{\pm}\mathrm{mes}_2\Omega_t^{\pm} + \left\{t\sum_{\pm}\gamma_1^{\pm}\mathrm{mes}_2\Omega_0^{\pm} + [\gamma_0]\int_{\Gamma_0} h(t;\,s)\,ds\right\} +$$
$$+\left\{t^2\sum_{\pm}\gamma_2^{\pm}\mathrm{mes}_2\Omega_0^{\pm} + \int_{\Gamma_0}\left(t[\gamma_1]h(t;\,s) + \frac{1}{2}[\gamma_0]\kappa(s)h(t;\,s)^2\right)ds\right\} + \cdots \tag{2.11}$$

In the calculation of (2.10) we have used the equality $2E_t = R_t$, which follows from definition (1.7) and Green's formula for the solution of problem (1.3)–(1.5), and in relation (2.11) the increments of the volumes of the phases $\mathrm{mes}_2(\Omega_t^{\pm}\backslash\Omega_0^{\pm}) - \mathrm{mes}_2(\Omega_0^{\pm}\backslash\Omega_t^{\pm})$ are written as a repeated integral. Moreover, we have introduced the notation $[\gamma_t] = \gamma_t^+ - \gamma_t^-$ etc. Note that the sum of the second terms from the right-hand sides of equalities (2.10) and (2.11) is identical with the component $tU^{(1)}(h)$ of trinomial (1.9), while the sum of the third terms is identical with the component $t^2U^{(1)}(h)$.

3. The fundamental and second terms of the asymptotic form

It is clear that we must take the solution of problems (1.3)–(1.5) with interface Γ_0 and load g^0 as the fundamental term u^0 of ansatz (2.6). Since the data of the asymptotic form can be exactly transferred from the perturbed contour Γ_t to the basic contour Γ_0, the second term u^1 is also the solution of the problem for the composite body Ω_0. The equilibrium Eq. (1.3) in $\Omega_0^{\pm}$, the boundary conditions (1.4) with right-hand side g^1 and the inhomogeneous interface conditions

$$[u_k^p] = \varphi_k^p, \quad [n_j\sigma_{kj}] = \psi_k^p, \quad k = 1, 2, \quad \text{on} \quad \Gamma_0 \tag{3.1}$$

in which $p = 1$, are satisfied for u^1.

We emphasise that here and henceforth all the jumps are calculated precisely on the arc Γ_0. The data on φ^1 and ψ^1 in conditions (3.1) are found from the requirement for expressions (2.7)–(2.9) to vanish. The sum from the first braces in relation (2.7) vanishes if $\varphi^1 = -[\partial_n u^0] = 0$, i.e. according to formulae (2.3) and the equation $[u^0] = 0$ we have

$$\begin{aligned}\varphi_n^1 &= -h[\partial_n u_n^0] = -h[\varepsilon_{nn}^0]\\ \varphi_s^1 &= -h[\partial_s u_s^0] = -2h[\varepsilon_{sn}^0] + hD^{-1}(\partial_s[u_n^0] - \kappa[u_s^0]) = -2h[\varepsilon_{sn}^0]\end{aligned} \tag{3.2}$$

Analysing expansions (2.8) and (2.9) we obtain

$$\psi_n^1 = -h[\partial_n\sigma_{nn}^0] + 2h'[\sigma_{ns}^0], \quad \psi_s^1 = -h[\partial_n\sigma_{ns}^0] - h'([\sigma_{nn}^0] - [\sigma_{ss}^0])$$

We will convert these functions using the equalities $D = 1$ and $[\sigma_{nn}^0] = [\sigma_{ns}^0] = 0$ on the contour Γ_0 and the equilibrium Eqs. (2.4) in $\Omega_0^{\pm} \cup \Gamma_0$ for the stresses $\sigma_{\alpha\beta}^0$:

$$\begin{aligned}\psi_n^1 &= h[\partial_s\sigma_{ns}^0] - h\kappa[\sigma_{ss}^0] + h\kappa[\sigma_{nn}^0] = -h\kappa[\sigma_{ss}^0]\\ \psi_s^1 &= h[\partial_s\sigma_{ss}^0] + 2h\kappa[\sigma_{ns}^0] + h'[\sigma_{ss}^0] = \partial_s(h[\sigma_{ss}^0])\end{aligned} \tag{3.3}$$

A unique solution of problem (1.3), (1.4), (3.1) with load g^1 and with jumps (3.2) and (3.3) exists at the interface Γ_0. This solution can be represented in the form of a sum

$$u^p = u^{p0} + u^{p1} \tag{3.4}$$

Here $p=1$ and u^{10} is the solution for zero jumps on Γ while u^{11} is the solution when there is no load g^1 on Σ. For the second integral of relations (2.10) we derive the equality

$$\int_\Sigma (g^0 \cdot u^1 + tg^1 \cdot u^0)ds = \int_\Sigma (g^0 \cdot u^{10} + tg^1 \cdot u^0)ds + \int_\Sigma g^0 \cdot u^{11} ds \tag{3.5}$$

The first term on the right is independent of the function h, which describes the perturbation of the contour. Integrating by parts, we convert the last integral on the right-hand side of (3.5), using relations (3.2), (3.3) and (2.3)

$$\begin{aligned}\int_\Sigma g^0 \cdot u^{11} ds &= \int_\Sigma \nu_j(\sigma^0_{kj}u^{11}_k - \sigma^{11}_{kj}u^0_k)ds = -\int_{\Gamma_0} n_j(\sigma^0_{kj}[u^{11}_k] - [\sigma^{11}_{kj}]u^0_k)ds = \\ &= \int_{\Gamma_0}(\sigma^0_{nn}\varphi^1_n + \sigma^0_{ns}\varphi^1_s - \psi^1_n u^0_n - \psi^1_s u^0_s)ds = \\ &= \int_{\Gamma_0}(h\sigma^0_{nn}[\varepsilon^0_{nn}] + 2h\sigma^0_{ns}[\varepsilon^0_{ns}] - h\kappa[\sigma^0_{ss}]u^0 + \partial_s(h[\sigma^0_{ss}])u^0_s)ds = \\ &= \int_{\Gamma_0} h(\sigma^0_{nn}[\varepsilon^0_{nn}] + 2\sigma^0_{ns}[\varepsilon^0_{ns}] - [\sigma^0_{ss}]\varepsilon^0_{ss})ds =: \int_{\Gamma_0} h(s)f_0(s)ds\end{aligned} \tag{3.6}$$

Here and henceforth the functions continuous on Γ_0 as a consequence of the matching conditions, will be written without the superscripts $\pm$.

According to the calculation of (2.11) and (3.5), (3.6) the sum

$$-\frac{1}{2}\int_{\Gamma_0} hf_0 ds + [\gamma^0]\int_{\Gamma_0} hds \tag{3.7}$$

is the first variation of functional (1.6) when $t=0$, i.e. for conditions (1.10) and for $u^{10}=0$. Since it is assumed that, at the instant $t=0$, the interface occupies the position Γ_0, variation (3.7) necessarily vanishes, and of course, in view of the arbitrariness of the test function h, the following equality is satisfied

$$\frac{1}{2}(\sigma^0_{nn}[\varepsilon^0_{nn}] + 2\sigma^0_{ns}[\varepsilon^0_{ns}] - [\sigma^0_{ss}]\varepsilon^0_{ss}) = [\gamma_0] \quad \text{on} \quad \Gamma_0 \tag{3.8}$$

Note that, when equality (3.8) breaks down, functional (1.9) necessarily has no minimum for small t, since expression (3.7) can be assigned any value $O(t)$, specified in advance, by appropriate choice of the function h.

Simple algebra shows that relation (3.8) is equivalent to the usual thermodynamic condition at the interface (see Refs. 1–3 etc.). We emphasise that, by virtue of the homogeneous conditions (1.5) on Γ_0, the stresses σ^0_{nn}, σ^0_{ns} and the strains ε^0_{ss} are continuous on the contour Γ_0, and requirement (3.8) can be formulated as follows:

$$\overline{\sigma^0} : [\varepsilon^0] - [\sigma^0] : \overline{\varepsilon^0} = 2[\gamma_0] \quad \text{on} \quad \Gamma_0$$

Here $[v]$ and $\bar{v} = (v^+ + v^-)/2$ are the jump and mean value of the function v on the contour Γ_0.

4. The third term of the asymptotic form

The term u^2 of ansatz (2.6) satisfies the interface conditions (3.1), where $p=2$, while the right-hand sides, by expansion (2.7), have the form

$$\begin{aligned}\varphi^2_n &= -h[\partial_n u^1_n] - \frac{1}{2}h^2[\partial^2_n u^0_n] = -h[\varepsilon^{11}_{nn}] - h^2\varphi^{21}_n - h\varphi^{20}_n \\ \varphi^2_s &= -h[\partial_n u^1_s] - \frac{1}{2}h^2[\partial^2_n u^0_s] = -2h[\varepsilon^{11}_{ns}] - h^2\varphi^{21}_s - h\varphi^{20}_s - hh'\Phi^{21}_s\end{aligned} \tag{4.1}$$

Here, by formulae (2.3), (3.2) and (3.4) and conditions (3.1) for $[u^1]$, the following equalities hold

$$\begin{aligned} \varphi_n^{21} &= \frac{1}{2}[\partial_n^2 u_n^0], \quad \varphi_s^{21} = \frac{1}{2}[\partial_n^2 u_s^0] + \partial_s[\varepsilon_{nn}^0] - 2\kappa[\varepsilon_{sn}^0] \\ \varphi_n^{20} &= t[\partial_n \hat{u}_n^1], \quad \varphi_s^{20} = t[\partial_n \hat{u}_s^1], \quad \Phi_s^{21} = [\varepsilon_{nn}^0] \end{aligned} \tag{4.2}$$

We emphasise that the functions (4.2) are independent of h and are determined from the solutions u^0 ands u^{10} of the problem for the composite body Ω_0 with zero jumps on Γ_0, while $\hat{u}^1$ is the solution of problem (1.3)–(1.5) for the body Ω_0 under the load $\hat{g} = g^1$. Using formulae (2.4), (3.3) and (3.4) and conditions (3.1) for the jumps $[\sigma_{nn}^1]$, $[\sigma_{ns}^1]$, we can convert the right-hand sides of the equalities $[\sigma_{nn}^2] = \ldots$, and $[\sigma_{ns}^2] = \ldots$, similar to (4.1), extracted from the right-hand sides of Eqs (2.8) and (2.9), and hence arrive at the relations

$$\begin{aligned} \psi_n^2 &= -\kappa h[\sigma_{ss}^{11}] + h^2\psi_n^{21} + h\psi_n^{20} + h'h\Psi_n^{21} + (h''h + h'^2)[\sigma_{ss}^0] \\ \psi_s^2 &= -\partial_s(h[\sigma_{ss}^{11}]) + h^2\psi_s^{21} + \partial_s(h\psi_s^{20}) + hh'\Psi_s^{21} \end{aligned} \tag{4.3}$$

Here, by the representations $u^{10} = t\hat{u}^1$ and $\sigma^{10} = t\hat{\sigma}^1$ the following quantities appear

$$\begin{aligned} \psi_n^{21} &= \partial_s^2[\sigma_{ss}^0] - \kappa^2[\sigma_{ss}^0] - \frac{1}{2}[\partial_n^2\sigma_{nn}^0], \quad \psi_s^{21} = 2\kappa\partial_s[\sigma_{ss}^0] - \frac{1}{2}[\partial_n^2\sigma_{ns}^0] \\ \psi_n^{20} &= -t\kappa[\hat{\sigma}_{ss}^1], \quad \psi_s^{20} = t[\hat{\sigma}_{ss}^1], \quad \Psi_n^{21} = 2\partial_s[\sigma_{ss}^0], \quad \Psi_s^{21} = \kappa[\sigma_{ss}^0] + [\partial_n\sigma_{ss}^0] \end{aligned} \tag{4.4}$$

In addition to the interface conditions (3.1) with the right-hand sides of (4.1) and (4.3) for the fields u^2 and σ^2 the equilibrium Eq. (1.3) and the boundary conditions (1.4) with the right-hand side g^2 from formula (1.1) are satisfied. A unique solution of this problem exists; it is continuous in $\Omega^\pm$ up to the contour Γ_0 and can be represented in the form of the sum (3.4) with $p=2$, where $u^{20} = t^2\hat{u}^2$ and u^{21} are the solutions of the problem in Ω_0 respectively with zero jumps on Γ_0 and zero external load on Σ.

Ignoring asymptotic terms that are independent of h in expansions (2.10) and (2.11), we can see that we need to analyse the following sum

$$-\frac{1}{2}\int_\Sigma (g^0 \cdot u^{21} + tg^1 \cdot u^{11})ds + \int_{\Gamma_0}\left(t[\gamma_1]h + \frac{1}{2}[\gamma_0]\kappa h^2\right)ds \tag{4.5}$$

which is a quantity $O(t^2)$ and the component functional $t^2(\mathcal{U}^{(2)}(h) - \mathcal{U}^{(2)}(0))$.

Repeating calculation (3.6) with obvious changes, we find that

$$\begin{aligned} \int_\Sigma g^1 \cdot u^{11} ds &= -\int_{\Gamma_0}(\sigma_{nn}^{10}\varphi_n^1 + \sigma_{ns}^{10}\varphi_s^1 - \psi_n^1 u_n^{10} - \psi_s^1 u_s^{10})ds = \\ &= t\int_{\Gamma_0} h(\hat{\sigma}_{nn}^1[\varepsilon_{nn}^0] + 2\hat{\sigma}_{ns}^1[\varepsilon_{ns}^0] - [\sigma_{ss}^0]\hat{\varepsilon}_{ss}^1)ds =: t\int_{\Gamma_0} h(s)f_1(s)ds \end{aligned} \tag{4.6}$$

We continue the calculations and obtain

$$\begin{aligned} \int_\Sigma g^0 \cdot u^{21} ds &= -\int_{\Gamma_0}(\sigma_{nn}^0\varphi_n^2 + \sigma_{ns}^0\varphi_s^2 - \psi_n^2 u_n^0 - \psi_s^2 u_s^0)ds = \\ &= \int_{\Gamma_0} h(s)(B(h; s) + b(s)h(s) + tf_2(s))ds \end{aligned} \tag{4.7}$$

Here

$$b = \varphi_n^{21}\sigma_{nn}^0 + \varphi_s^{21}\sigma_{ns}^0 + \psi_n^{21}u_n^0 + \psi_s^{21}u_s^0 - \frac{1}{2}\frac{\partial}{\partial s}(\Phi_s^{21}\sigma_{ns}^0 + \Psi_n^{21}u_n^0 + \Psi_s^{21}u_s^0) + \frac{1}{2}\frac{\partial^2}{\partial s^2}([\sigma_{ss}^0]u_s^0)$$
$$f_2 = \sigma_{nn}^0[\hat{\varepsilon}_{nn}^1] + 2\sigma_{ns}^0[\hat{\varepsilon}_{ns}^1] - \varepsilon_{ss}^0[\hat{\sigma}_{nn}^1] \tag{4.8}$$

and B is an operator, defined by the formula

$$\int_{\Gamma_0} H(s)B(h;\, s)ds = \int_{\Gamma_0} H(\sigma_{nn}^0[\varepsilon_{nn}^{11}] + 2\sigma_{ns}^0[\varepsilon_{ns}^{11}] - \varepsilon_{ss}^0[\sigma_{nn}^{11}])ds \tag{4.9}$$

in which H is the test function from $C^\infty(\Gamma_0)$, and the jumps $[\sigma_{\dots}^{11}]$ and $[\varepsilon_{\dots}^{11}]$, by definitions (3.2) and (3.3), are the values of the linear operators on h. Consequently, the following estimate holds

$$\sum_{\pm}(\|\varepsilon^{11\pm};\, H^{l-1/2}(\Gamma_0)\| + \|\sigma^{11\pm};\, H^{l-1/2}(\Gamma_0)\|) \le$$

$$\le c\sum_{\pm}\|u^{11\pm};\, H^{l+1}(\Omega_0^\pm \cap \mathcal{V})\| \le c\|h;\, H^{l+1/2}(\Gamma_0)\|$$

The constant c depends on the fields $\sigma^{0\pm}|_{\Gamma_0}$ and $\varepsilon^{0\pm}|_{\Gamma_0}$, Hooke's tensors $A^\pm$ and the geometry of the boundaries $\partial\Omega$ and Γ_0.

Thus, the operator B performs the continuous mapping: $H^{l+1/2}(\Gamma_0) \to H^{l-1/2}(\Gamma_0)$, and $H^m(\Gamma_0)$ implies a Sobolev-Slobodetskii space.

Now, by calculating and equating to zero the variation of the functional (4.5), we arrive at the following equation on the contour Γ_0

$$-\frac{1}{2}(B(h;\, s) + B^*(h;\, s)) + ([\gamma_0]\kappa(s) - b(s))h(s) = \frac{1}{2}t(f_1(s) + f_2(s) - 2[\gamma_1]),\ \ s \in \Gamma_0 \tag{4.10}$$

Its solution must be sought in the form (2.5) without the dots. Eq. (4.10) is called a growth equation, since its solution, in accordance with formula (2.1), gives information on the quasistatic development of the interface.

5. The operator *B*

In order to construct the operator (4.9) and the conjugate operator B^*, occurring in Eq. (4.10), we require new notation, which, at first glance, seems strange (see Section 7 for an explanation). We will define the columns

$$\zeta = (\sigma_{nn}, 2^{1/2}\sigma_{ns}, -\varepsilon_{ss})^\top, \quad \eta = (\varepsilon_{nn}, 2^{1/2}\varepsilon_{ns}, \sigma_{ss})^\top \tag{5.1}$$

where 2½ is a normalizing factor and the superscript T is the sign of transposition. Note that the column ζ^0, calculated from the solution u^0 of problem (1.3)–(1.5) for a fixed interface, is continuous on the contour Γ_0, while the thermodynamic condition (3.8) can now be formulated as

$$\frac{1}{2}[\eta^0]^\top\zeta^0 = [\gamma_0] \text{ on } \Gamma_0 \tag{5.2}$$

Using Hooke's law (1.2) for the stress and strain tensors in curvilinear coordinates, we obtain the relation between the columns (5.1) on Γ_0 and the side of the phase $\Omega_0^\pm$

$$\eta^\pm = Q^\pm\zeta^\pm \tag{5.3}$$

The matrix functions $Q^\pm$, continuous on the contour Γ_0, are symmetrical and reversible but are not sign-definite due to displacement of the positions σ_{ss} and ε_{ss} in formulae (5.1) (compare with the discussion in Section 7).

Further, the Steklov-Poincaré operators $S^\pm$ are necessary for elastic bodies $\Omega^\pm$. By specifying the displacement columns $\tilde{v}^\pm = (v_n^\pm, 2^{-1/2}v_s^\pm)^\top$ on Γ_0 they are made to correspond to the columns of the normal stresses

$\tilde{\zeta}^{\pm} = (\sigma_{nn}^{\pm}, 2^{1/2}\sigma_{ns}^{\pm})^{\top}$, taken on Γ_0 and obtained from the solutions w^+ and w^- of the homogeneous equilibrium equations in $\Omega^{\pm}$ with the same sets of boundary conditions:

$$w_n^+ = v_n^+, \quad w_s^+ = v_s^+ \quad \text{on} \quad \Gamma_0$$

$$w_n^- = v_n^-, \quad w_s^+ = v_s^+ \text{ on } \Gamma_0, \quad w^- = 0 \text{ on } \Upsilon, \quad n_j\sigma_{kj}^-(w^{\pm}) = 0, \quad k = 1, 2, \text{ on } \Sigma$$

It is well known (see, for example, Refs. 13,14), that the mapping

$$H^{l+1/2}(\Gamma_0)^2 \ni \tilde{v}^{\pm} \mapsto S^{\pm}\tilde{v}^{\pm} = \tilde{\zeta}^{\pm} \in H^{l-1/2}(\Gamma_0)^2 \tag{5.4}$$

is an isomorphism for any $l \geq 0$ and is a classical elliptic pseudodifferential operator of the first order. Like the unbounded operators in $L_2(\Gamma_0)^2$ (compare mappings (5.4) for $l = ½$) the operators $\pm S^{\pm}$ are closed self-conjugate and positive. The last two properties arise, for example, from Green's formulae

$$\int_{\Gamma_0} \tilde{\zeta}^{\pm}(w^{\pm})^{\top}\tilde{v}^{\pm}ds = \int_{\Gamma_0}(\sigma_{nn}^{\pm}(w^{\pm})v_n^{\pm} + \sigma_{ns}^{\pm}(w^{\pm})v_s^{\pm})ds = \pm\int_{\Omega^{\pm}}\sigma^{\pm}(w^{\pm}) : \varepsilon^{\pm}(v^{\pm})dx$$

We mean by $\varepsilon^{\pm}(v^{\pm})$, $\sigma^{\pm}(w^{\pm})$ etc. the strain and stress fields obtained from the displacement vectors $v^{\pm}$ and $w^{\pm}$ in $\Omega_0^{\pm}$.

By virtue of relations (3.2) and (3.3), the interface conditions (3.1) for the fields $u^{11\pm}$ and $\sigma^{11\pm}$ in the new notation become

$$[\tilde{u}^{11}] = -h[\tilde{\eta}^0], \quad [\tilde{\zeta}^{11}] = -\delta^* h[\eta_{ss}^0] \text{ on } \Gamma_0 \tag{5.5}$$

Here $\tilde{\eta} = (\varepsilon_{nn}, 2^{1/2}\varepsilon_{ns})^{\top}$ while δ and δ^* are formally conjugate differential operators acting in accordance with the equalities

$$\delta\tilde{u} = \kappa u_n + \partial_s u_s = \varepsilon_{ss}, \quad \delta^*\xi = (\kappa\xi, -2^{1/2}\partial_s\xi)^{\top} \tag{5.6}$$

Bearing formulae (5.5) and (5.4) in mind, we successively obtain

$$\tilde{u}^{11+} + \frac{1}{2}h[\tilde{\eta}^0] = \tilde{u}^{11-} - \frac{1}{2}h[\tilde{\eta}^0] =: \tilde{w}, \quad \tilde{\zeta}^{11+} = S^{\pm}\left(\tilde{w} \mp \frac{1}{2}h[\tilde{\eta}^0]\right)$$

$$-\delta^* h[\tilde{\eta}_{ss}^0] = [\tilde{\zeta}^{11}] = (S^+ - S^-)\tilde{w} - \frac{1}{2}(S^+ + S^-)h[\tilde{\eta}^0]$$

We put

$$S_\bullet = S^+ - S^-, \quad S_\# = \frac{1}{2}(S^+ + S^-)$$

Both operators are symmetrical and, moreover, $S_\bullet$ is positive and invertible (see the commentaries to formula (5.4)). Since

$$S_\bullet^{-1}S_\# \mp \frac{1}{2} = S_\bullet^{-1}S^{\mp}$$

taking definitions (5.6) into account we obtain that

$$\tilde{u}^{11+} = S_\bullet^{-1}S^{\mp}h[\tilde{\eta}^0] - S_\bullet^{-1}\delta^* h[\eta_{ss}], \quad \tilde{\zeta}^{11\pm} = S^{\pm}S_\bullet^{-1}S^{\mp}h[\tilde{\eta}^0] - S^{\pm}S_\bullet^{-1}\delta^* h[\eta_{ss}]$$

$$\zeta_{ss}^{11\pm} = -\varepsilon_{ss}^{11\pm} = -\delta S_\bullet^{-1}S^{\mp}h[\tilde{\eta}^0] + \delta S_\bullet^{-1}\delta^* h[\eta_{ss}]$$

We will write the last two equations in matrix form

$$\left\| \begin{matrix} \tilde{\zeta}^{11\pm} \\ \zeta_{ss}^{11\pm} \end{matrix} \right\| = N^{\pm} \left\| \begin{matrix} h[\tilde{\eta}^0] \\ h[\eta_{ss}] \end{matrix} \right\|, \quad N^{\pm} = \left\| \begin{matrix} S^{\pm}S_{\bullet}^{-1}S^{\mp} & -S^{\pm}S_{\bullet}^{-1}\delta^* \\ -\delta S_{\bullet}^{-1}S^{\mp} & \delta S_{\bullet}^{-1}\delta^* \end{matrix} \right\| = \left\| \begin{matrix} S^{\pm} \\ -\delta \end{matrix} \right\| S_{\bullet}^{-1}(S^{\mp}, -\delta^*) \tag{5.7}$$

and note that the operators $N^{\pm}$ are mutually conjugate. By virtue of relation (5.3) we have

$$[\eta^0] = (Q^+ - Q^-)\zeta^0, \quad [\eta^{11}] = (Q^+N^+ - Q^-N^-)(Q^+ - Q^-)\zeta^0 h \tag{5.8}$$

Substituting relations (5.8) into the integral identity (4.9), we find that

$$B = (\zeta^0)^{\top}(Q^+N^+ - Q^-N^-)(Q^+ - Q^-)\zeta^0 \tag{5.9}$$

Consequently

$$B + B^* = (\eta^{0+})^{\top}(N^+ + N^-)\eta^{0+} + (\eta^{0-})^{\top}(N^+ + N^-)\eta^{0-} - (\eta^{0-})^{\top}N^+\eta^{0-} - (\eta^{0-})^{\top}N^-\eta^{0+} \text{ (5.10)} \tag{5.10}$$

Note that the operators ((5.9) and (5.10) act on the function h as follows: initially the expressions $\eta^{0\pm} = Q^{\pm}\zeta^0$ are multiplied by h and only then are the operators $N^{\pm}$ employed.

Despite the symmetrical structure, the operator $B + B^*$, generally speaking, is not sign-definite. The point is that after introducing the column $Y = (Y^{-+}, Y^{++}, Y^{--}, Y^{+-})^{\top}$ with the scalar components

$$Y^{\alpha\beta} = (S_{\bullet}^{-1/2}S^{\alpha}, -S_{\bullet}^{-1/2}\delta^*)\eta^{0\beta}, \quad \alpha, \beta = +, - \tag{5.11}$$

in accordance with equalities (5.7), the sum (5.10) takes the form

$$B + B^* = (Y)^{\top}TY \tag{5.12}$$

Here $S_{\bullet}^{-1/2}$ is the positive root of the operator $S_{\bullet}^{-1}$ and T is a numerical 4×4 matrix with the same eigencolumns X^j and eigenvalues Λ_j

$$T = \left\| \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{matrix} \right\|, \quad X^j = \left\| \begin{matrix} \Lambda_j(\Lambda_j^2 - 2) \\ 1 - \Lambda_j^2 \\ \Lambda_j \\ 1 \end{matrix} \right\|, \quad j = 1, \dots, 4$$

$$\Lambda_1 = -\Lambda_3 = \sqrt{\frac{3 + \sqrt{5}}{2}}, \quad \Lambda_2 = -\Lambda_4 = \sqrt{\frac{3 - \sqrt{5}}{2}}$$

The presence of the pairs of positive eigenvalues Λ_1 and Λ_2 and negative eigenvalues Λ_3 and Λ_4 denotes that the operator (5.12) becomes positive or negative only for certain relations between the components (5.11). Nevertheless, formula (5.12) is useful for calculating the ellipticity of the operator $B + B^*$. Really by denoting the principal symbols of the operators $S^{\pm}$ etc. by $\mathbf{S}^{\pm}(s; \xi)$ etc. at the point $s \in \Gamma_0$, we obtain that

$$\mathbf{B}(s; \xi) + \mathbf{B}^*(s; \xi) = \sum_{j=1}^{2} (1 + \Lambda_j^2)(|\mathbf{C}_j(s; \xi)|^2 - |\mathbf{C}_{j+2}(s; \xi)|^2) \tag{5.13}$$

where $\boldsymbol{C}_j(s; \xi)$ are the coefficients of the expansion

$$\mathbf{Y}(s; \xi) = \sum_{j=1}^{4} X^j \mathbf{C}_j(s; \xi), \tag{5.14}$$

while the principal symbol (5.14) of the operator (5.11) is equal to (i is the square root of -1)

$$(\mathbf{S}_{\bullet}(s;\xi)^{-1/2}\mathbf{S}^{\alpha}(s;\xi), \mathbf{S}_{\bullet}(s;\xi)^{-1/2}(0, 2^{1/2}i\xi)^{\top})\eta^{0\pm}(s)$$

Hence, the first-order operator (5.9) is elliptic on Γ_0 if and only if expression (5.13) does not vanish for all $\xi \in \mathbb{R}\backslash\{0\}$ and $s \in \Gamma_0$.

Unfortunately we were unable to invoke thermodynamic conditions (5.2) or (3.8) to investigate the properties of the operators B or $B+B^*$. The sole useful but trivial conclusion is that the columns ζ^0 and Y in formulae (5.2) and (5.12) cannot be zero.

6. The stability of the quasistatic process

Several different situations may arise when solving Eq. (4.10).

1°. The growth equation has a unique continuous solution, which makes the functional $\mathcal{U}^{(2)}(h)$ reach a global minimum.
2°. There is a family of continuous solutions, each of which makes the functional $\mathcal{U}^{(2)}(h)$ reach a local minimum.
3°. Continuous solutions exist, but at corresponding stationary points the functional $\mathcal{U}^{(2)}(h)$ does not reach minimum values.
4°. The growth equation has no continuous solutions.

Case 1°, in which a stable quasistatic evolution of the interface Γ_t occurs, arises for the elliptic operator (4.10) and for the following quadratic form that is positive-definite in the space $H^{1/2}(\Gamma_0) \times H^{1/2}(\Gamma_0)$

$$\mathbf{q}(h, H) := -\frac{1}{4}\int_{\Gamma_0}\{H(s)B(h;s) + h(s)B(H;s) + 2(b(s) - [\gamma_0]\kappa(s))H(s)h(s)\}ds \tag{6.1}$$

The integral is understood as the duality between the spaces $H^{1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0) = H^{1/2}(\Gamma_0)^*$.

Suppose now that the symbol (5.13) is negative, but the homogeneous growth equation has non-trivial solutions; in particular the form (6.1) is not positive. According to the Fredholm alternative, the solution h of Eq.(4.10) only exists if the following orthogonality conditions hold

$$\int_{\Gamma_0} h^{(j)}(s)(f_1(s) + f_2(s) - [\gamma_1])ds = 0, \quad j = 1, \ldots, J \tag{6.2}$$

where $h^{(1)}, \ldots, h^{(j)}$ is the basis in the lineal $\mathcal{L}$ of the solutions of the homogeneous equation. If requirement (6.2) is violated, situation 4° occurs, the quasistatic process becomes dynamic, and one is not justified in ignoring inertial terms; in this case the invoking of the lower terms of the asymptotic expansions does not improve the situation, since the refined growth equation becomes singularly perturbed (it contains a small parameter t in the higher derivative), while the norm of the solution $\hat{h}$ becomes infinitely large as $t \to +0$ (ansatz (2.5) breaks down). If the conditions for (6.2) to be orthogonal are satisfied, then, within the framework of situation 2° we must speak of possible bifurcations of the interface (the solution h is determined apart from a term of the lineal $\mathcal{L}$), but an investigation of the bifurcations and the appearance among them of stable ones is not limited by an investigation of form (6.1) and requires additional considerations.

The operator $B+B^*$ may be elliptic, and form (6.1) may be negative-definite. In other words, as previously, a unique solution of Eq. (4.10) exists, but it makes the functional $\mathcal{U}^{(2)}(h)$ reach a maximum, i.e. we have case 3°, in which all the positions of the inteface Γ_t are in unstable equilibrium, and on changing from Γ_0 to Γ_t, $t > 0$, the quasistatic process may spontaneously change into a dynamic one.

Eq. (4.10) may remain uniquely solvable even when the symbol (5.13) vanishes at several points of the contour Γ_0. However, in this case the solution ceases to be continuous (compare with situation 4°), and the asymptotic analysis becomes perhaps only formal (compare with Section 7 (1)). As usual, the breakdown of the asymptotic structures due to the impossibility of establishing the smallness of the residues is related to attempts to give a quasistatic description to a particularly dynamic process.

We will digress from considering the quasistatic evolution of the interface, use limitations (1.10) and obtain the functional $\mathcal{U}$ from formula (1.6) when $t=0$. As already mentioned in Section 3, the first variation of this functional, due to a small perturbation (2.1) of the contour Γ_0, has the form (3.7). It is easy to check that the second variation is identical with the integral

$$-\frac{1}{2}\int_{\Gamma_0} h(s)(B(h;s)+b(s)h(s)-[\gamma_0]\kappa(s)h(s))ds \tag{6.3}$$

Hence, for the classical stability of the position Γ_0 of the interface it is necessary for the quadratic form (6.1) to be positive-definite.

7. Discussion

7.1. The requirements regarding the smoothness of the interface

The method of transferring data from a regularly perturbed boundary to a base, widely used in the mechanics of curvilinear cracks (see Refs. 15–17 etc.), found justification in the book.[18] It requires greater smoothness of the boundary than the apparatus of material derivatives (see Refs. 11,2,12 etc.), but leads to clear and fairly simple formulae, since canonical objects: Steklov-Poincaré operators, tangential gradients, curvatures, etc. are used, rather than "almost identical" diffeomorphisms, always defined with considerable arbitrariness.

To ensure inclusion $h\in H^{l+1/2}(\Gamma_0)$ it is necessary that $u^{0\pm}\in H^{l+3}(\Omega_0^{\pm})^2$ and $\sigma^{\pm 0}\in H^{l+2}(\Omega_0^{\pm})^{2\times 2}$. Here the contour Γ_0 must be of the class C^{l+2}. If $l\geq 2$, then, using general results[18] one can establish that the $H^{l-1}(\Omega_t)^2$-norm of the residue in representation (2.6) is estimated by the quantity ct^4.

We emphasise that, when using the equilibrium Eq. (2.4), from the final formulae one must eliminate many normal derivatives of the elastic fields, which happens to be useful for computational schemes, for example, in the method of boundary integral equations. This was done in transformations (3.3), but in expressions (4.2) and (4.4) and later the second normal derivatives of the displacements u_n and u_s remained without processing because, for arbitrary anisotropy, extremely complicated expressions are obtained. The derivatives $\partial_n\sigma_{ss}^{0\pm}$ are not eliminated from the final expressions.

7.2. The surface enthalpy

Scalar products of the columns (5.1) appeared in many formulae – see, for example, (3.8), (4.6), (4.9) and (4.8). Definition (5.1) has a physical background. The stresses σ_{nn}, $\sigma_{ns}=\sigma_{sn}$ and the strains ε_{ss} at the interface among the components of the stress and strain tensors turn out to be constrained – they cannot be assumed to be arbitrary due to the interface conditions (1.5), and hence, instead of the density W of the elastic energy it is natural to take the state function Π, for which

$$\frac{\partial\Pi}{\partial\varepsilon_{nn}}=\sigma_{nn},\quad \frac{\partial\Pi}{\partial\varepsilon_{ns}}=\sigma_{ns},\quad \frac{\partial\Pi}{\partial\varepsilon_{sn}}=\sigma_{sn},\quad \frac{\partial\Pi}{\partial\sigma_{ss}}=-\varepsilon_{ss},\quad \text{or}\quad \frac{\partial\Pi}{\partial\eta}=\zeta \tag{7.1}$$

The components of the column η are free at the interface. It is not difficult to obtain the required function from the conservation of the linear Hooke's law:

$$\Pi = W-\sigma_{ss}\varepsilon_{ss} = \frac{1}{2}\sigma:\varepsilon-\sigma_{ss}\varepsilon_{ss} = \frac{1}{2}(Q\eta)^{\top}\eta \tag{7.2}$$

Since the change from W to Π is made using the same rule as the change from energy to enthalpy, we can somewhat freely call the form (7.2) the surface enthalpy.

When setting up the columns ζ and η the minus sign, which occurs in relations (5.3) and which ensures that the matrix Q is symmetrical in relation (5.3), and the equations $\sigma_{ns}=\sigma_{sn}$, $\varepsilon_{ns}=\varepsilon_{sn}$ are taken into account. It is easy to show that the left-hand side of thermodynamic condition (5.2) is the jump $[\Pi]=\Pi^{+}-\Pi^{-}$.

7.3. The surface energy

We will assume that the position of the interface Γ_t is determined by minimizing the functional $\mathcal{U}_t + \mathcal{H}_t$, where the term $\mathcal{U}_t$ has already been indicated in relations (1.6)–(1.8), while the term $\mathcal{H}_t$ is the surface energy, distributed along Γ_t with constant density $\beta > 0$,

$$\begin{aligned}\mathcal{H}_t &= \beta \operatorname{mes}_1 \Gamma_t = \beta \int_{\Gamma_0} (1 + |\partial_s h(t;s)|^2)^{1/2} (1 + h(t;s)\kappa(s)) ds = \\ &= \beta \operatorname{mes}_1 \Gamma_0 + \beta \int_{\Gamma_0} h\kappa ds + \frac{\beta}{2} \int_{\Gamma_0} |\partial_s h|^2 ds + \dots\end{aligned} \tag{7.3}$$

We will denote the length of the arc Γ_t by $\operatorname{mes}_1 \Gamma_t$. The addition to integrals (3.7) and (4.7) of terms from the right-hand side of Eq. (7.3) leads to the following changes in the mathematical description of the quasistatic development of the interface: the thermodynamic condition (3.8) takes the form

$$\frac{1}{2}(\sigma_{nn}^0[\varepsilon_{nn}^0] + 2\sigma_{ns}^0[\varepsilon_{ns}^0] - [\sigma_{ss}^0]\varepsilon_{ss}^0) = [\gamma_0] + \beta\kappa \quad \text{on} \quad \Gamma_0$$

while the growth Eq. (4.10) converts to the following

$$-\beta\partial_s^2 \hat{h} - B(\hat{h}) - B^*(\hat{h}) + 2[\gamma_0]\kappa\hat{h} - 2b\hat{h} = f_1 + f_2 - 2[\gamma_1] \quad \text{on} \quad \Gamma_0 \tag{7.4}$$

The common factor $t/2$ has been shortened, and we have borne in mind representation (2.5). The occurrence of the second derivative in integrodifferential Eq. (7.4) facilitates the stability, discussed in Section 6. In particular, any solution $h \in L_2(\Gamma_0)$ of Eq. (7.4) turns out to be continuous everywhere on Γ_0 and hence situation 4°, which requires a dynamic formulation of the problem, is only possible when there are non-trivial solutions of Eq. (7.4) and the corresponding orthogonality conditions (6.2) break down.

7.4. The non-locality of the stability condition

Representation (5.9) of the operator B, which occurs in expression (6.3) for the second variation of functional (1.6) when $t = 0$, contains the Steklov-Poincaré operators (5.4), that are integro-differential and non-local in nature. In particular, only the principal symbol (5.12) can be calculated for known Hooke tensors $A^{\pm}$, the interface Γ_0 and the solution u^0, since the compact components of the operator $B + B^*$ depend on the global characteristics: the shape of the body Ω and the position of the arc γ, to which it is restrained. We will carry out the following mental experiment: we will cut out from the body Ω a section Ω', which does not intersect the subregion Ω_0^+, and we will apply to the newly formed boundary the same normal forces $n_j\sigma_{kj}^{0\pm}$ which arose in the body $\Omega_0^+ \cup \Omega_0^-$ under the load g^0. The elastic fields do not change in the remaining part $\Omega_0^+ \cup (\Omega_0^- \backslash \Omega')$, i.e. the thermodynamic condition (3.8) (or (5.2)) remains the same. At the same time the change in the operator B and the second variation (6.3), due to perturbation of the outer boundary of the body, may radically affect the stability of the interface.

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